atmosphere has been exited but before a significant part of orbital velocity has been achieved, values of ψ in the range 30–45 deg are appropriate, rather than the launch value of 90 deg. These then are the values that should be used in Eq. (18).

In a similar manner it is possible to treat the case at the instant orbital conditions are achieved. The principal deviation from the previous is that now we must have $\dot{\psi} = \eta - \dot{\theta} = \eta + \omega$. The solution for this is shown in Ref. 14. For η to be constant, the same value must apply to both the launch and orbital end conditions; graphical analysis then shows that only ηs in the narrow range of about $-0.7\ \omega$ to $-1.0\ \omega$ are admissible. From the equation

$$\Delta\psi + \Delta\theta = \eta T \tag{19}$$

we can estimate the thrusting time T. (The $\Delta\theta$ is the central angle traversed during the maneuver and is approximately equal to $\omega T/2$. The $\Delta\psi$ is the difference between the final and initial pitch angles in the local frame, and is equal to about 30 deg when $\eta = -\omega$. Both Δ are negative, but then so is η according to our convention.) The estimate produced by Eq. (19) is about 900 s, which is reasonable but on the high side.

Conclusion

Actual launch vehicles have a higher acceleration profile and are able to achieve orbit in about 600 s. Consequently their pitch-rate magnitude is more in the range 1.5 to 2 ω, and falls out of the previous range. Thus, while exactly linear pitch profiles probably do not apply, the above analysis suggests why a nearly constant pitch rate is tenable. (See also Ref. 14.)

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Invariant Set Analysis of the Hub-Appendage Problem

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Introduction

In the recent literature, an asymptotic stability theorem[1] for autonomous and periodic nonautonomous systems was used to prove the global asymptotic stability of the mass-spring-damper system and the damped Mathieu system. For such systems, the application of LaSalle's invariant set theorem[3] has been the conventional approach adopted to prove the global asymptotic stability. When the derivative of the Lyapunov function[2] vanishes, LaSalle's theorem[3] requires us to show that the maximum invariant set of the system consists only of the equilibrium point at its entry. Although it is always simple to identify the set of points Q where the derivative of the Lyapunov function vanishes, the maximum invariant set $I \subset Q$ is not always easy to identify. The main challenge of LaSalle's theorem[3] is therefore to sort out the maximum invariant set. For a distributed parameter system the dynamics are described by a hybrid set of ordinary and partial differential equations. For such a system, the sorting out of the maximum invariant set is not a trivial task. In such a situation it is useful to apply the theorem in Ref. 1 so as to comment on the asymptotic stability of the system.

The distributed parameter system consisting of a rigid hub with one or more cantilevered flexible appendages has appeared in the technical literature quite frequently (see Refs. 4, 5, 6, and 7). The system described in Fig. 1 consists of four appendages that are identical uniform beams conforming to the Euler-Bernoulli assumptions. Each beam cantilevered rigidly to the hub is assumed to have a tip mass. The motion of the system is confined to the horizontal plane and the control torque is generated by a single-reaction wheel actuator. Under the assumption that the system undergoes antisymmetric motion with deformation in unison (see Fig. 2), a class of rest-to-rest maneuvers was considered in Ref. 4. For the particular Lyapunov function considered, the best choice of the control input only guaranteed the negative semidefiniteness of the derivative of the Lyapunov function. To conclude the global asymptotic stability using LaSalle's theorem, it would be necessary to formally prove that the maximum invariant set consists only of the equilibrium point. The global asymptotic stability of the system was claimed in Ref. 4 in the absence of this proof.

In this Note we consider the hub-appendage problem[4] with modifications. The modeling and successful control of such a system is expected to provide us with insight into the modeling and control of a general class of distributed parameter systems. Using a Lyapunov function approach and the asymptotic stability theorem in Ref. 1, we prove that global asymptotic stability of the system is guaranteed provided the system undergoes antisymmetric motion with deformation in unison.

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In other situations, such as symmetric motion with deformation in opposition (see Fig. 2), such a conclusion cannot be drawn.

Theorem on Asymptotic Stability

Consider the nonautonomous system

$$\dot{x} = f\Big((t, x(t)\Big) \tag{1}$$

where $f{:}R_+ \times D \rightarrow R^n$ is a smooth vector field on $R_+ \times D$, $D \subset R^n$ in the neighborhood of the origin $x = 0$. Let $x = 0$ be an equilibrium point for the system described by Eq. (1). We now state the theorem on asymptotic stability.[1]

Theorem. 1) A necessary condition for stable nonautonomous systems: Let $V(t, x) : R_+ \times D \rightarrow R_+$ be locally positive definite and analytic on $R_+ \times D$, such that

$$\dot{V}(t, x) \triangleq \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right) f(t, x)$$

is locally negative semidefinite. Then whenever an odd derivative of V vanishes, the next derivative necessarily vanishes and the second next derivative is necessarily negative semidefinite. 2) A sufficient condition for asymptotically stable autonomous systems: Let $V(x) : D \rightarrow R_+$ be locally positive definite and analytic on D, such that $\dot{V} \leq 0$. If there exists a positive integer k such that

$$\begin{cases} V^{(2k+1)}(x) < 0 & \forall x \neq 0 : \dot{V}(x) = 0 \\ V^{(i)}(x) = 0 & \text{for } i = 2, 3, \ldots, 2k \end{cases} \tag{2}$$

where $V^{(*)}(x)$ denotes the $(*)$th time derivative of V with respect to time, then the system is asymptotically stable. However, if $V^{(j)}(x) = 0$, $\forall j = 1, 2, \ldots, \infty$, then the sufficient condition for the autonomous system to be asymptotically stable is that the set

$$S = \{x : V^{(j)}(x) = 0, \quad \forall j = 1, 2, \ldots, \infty\}$$

contains only the trivial trajectory $x = 0$.

Hub-Appendage Problem

This example is taken from Ref. 4 with some modifications. The hybrid system of ordinary and partial differential equations governing the dynamics of the system, which has already been described in the introduction, is

$$I_{\text{hub}} \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = u + \sum_{i=1}^{4} (M_{i0} - rS_{i0}) \tag{3}$$

$$-(M_{i0} - rS_{i0}) = \int_r^l \rho x \left(\frac{\partial^2 y_i}{\partial t^2} + x \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) \mathrm{d}x + ml \left(l \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\partial^2 y_i}{\partial t^2}\bigg|_l\right)$$

$$i = 1, 2, 3, 4 \tag{4}$$

$$\rho\left(\frac{\partial^2 y_i}{\partial t^2} + x \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + EI \frac{\partial^4 y_i}{\partial x^4} = 0, \qquad i = 1, 2, 3, 4 \tag{5}$$

The boundary conditions on Eqs. (3-5) are

$$y_i(t, r) = \frac{\partial y_i}{\partial x}\bigg|_r = 0, \qquad i = 1, 2, 3, 4 \tag{6}$$

$$\frac{\partial^2 y_i}{\partial x^2}\bigg|_l = 0, \qquad i = 1, 2, 3, 4 \tag{7}$$

$$\frac{\partial^3 y_i}{\partial x^3}\bigg|_l = \frac{m}{EI}\left(l \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\partial^2 y_i}{\partial t^2}\bigg|_l\right), \qquad i = 1, 2, 3, 4 \tag{8}$$

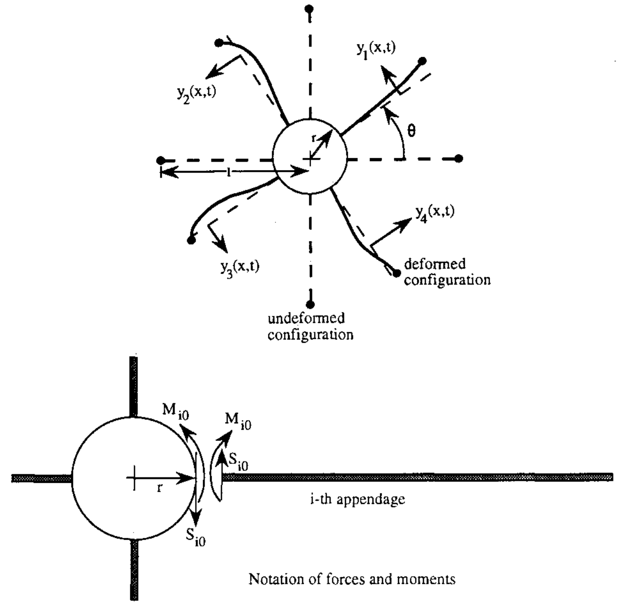

Fig. 1 Distributed parameter autonomous system consisting of a rigid hub with four cantilevered flexible appendages.

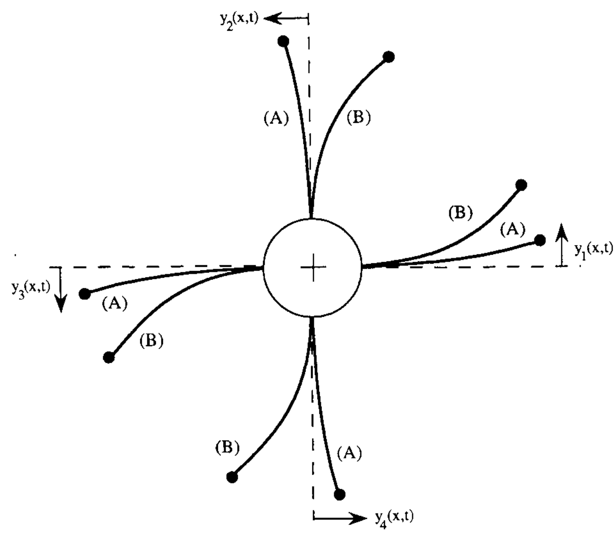

Fig. 2 Antisymmetric and symmetric motion of the system consisting of a rigid hub and four flexible appendages: A is the antisymmetric motion (deformation in unison), $y_1(x, t) = y_2(x, t) = y_3(x, t) = y_4(x, t)$ and B is the symmetric motion (deformation in opposition), $y_1(x, t) = -y_2(x, t)$, $y_3(x, t) = -y_4(x, t)$.

The state of the system is described by a hybrid set of discrete and continuous variables:

$$Z = \left[\theta, \dot{\theta}, y_1(x, t), \ldots, y_4(x, t), \frac{\partial y_1(x, t)}{\partial t}, \ldots, \frac{\partial y_4(x, t)}{\partial t}\right] \tag{9}$$

We choose the Lyapunov function V as

$$V = \frac{a_1}{2} I_{\text{hub}} \dot{\theta}^2 + \frac{a_2}{2}(\theta - \theta_f)^2 + \frac{a_3}{2} \sum_{i=1}^{4} \left[\int_r^l \rho\left(\frac{\partial y_i}{\partial t} + x\dot{\theta}\right)^2 \mathrm{d}x\right.$$

$$\left. + \int_r^l EI\left(\frac{\partial^2 y_i}{\partial x^2}\right)^2 \mathrm{d}x + m\left(l\dot{\theta} + \frac{\partial y_i}{\partial t}\bigg|_l\right)^2\right] \tag{10}$$

to derive control laws that will drive the system to its desired state $Z_{\text{desired}} = (\theta_f, 0, 0, \ldots, 0, 0, \ldots, 0)$. In Eq. (10), a_1, a_2, and

a_3 are positive constants. It can be shown[4] that the choice of $u(t)$ as

$$u = -(1/a_1)\left[a_2(\theta-\theta_f)+a_4\dot{\theta}+(a_3-a_1)\sum_{i=1}^{4}(rS_{i0}-M_{i0})\right] \qquad a_4>0 \tag{11}$$

in Eq. (3), leads to $\dot{V}=-a_4\dot{\theta}^2$. Clearly, $\dot{V}$ is negative semidefinite and is equal to zero if $\dot{\theta}=0$. To check for the asymptotic stability of the system using the theorem in Ref. 1, we first compute the higher-order derivatives of V. We find that when $\dot{V}=0$, the following always holds

$$V^{(2k+1)} = -2^k a_4[\theta^{(k+1)}]^2, \quad V^{(i)}=0, \quad i=1,2,\ldots,2k \tag{12}$$

for some positive integer k. In Eq. (12), $V^{(*)}$ denotes the $(*)$th time derivative of V, and $\theta^{(*)}$ denotes the $(*)$th time derivative of θ. Using Eq. (12) and the sufficient conditions of the asymptotic stability theorem,[1] we conclude that the system is globally asymptotically stable if $\theta^{(k)}\neq 0$ for any positive integer k. In other words, if $\dot{V}=0$ at some time $t=T$, then the system will be globally asymptotically stable if θ is not a constant for all $t\geq T$, and is a constant only at the equilibrium point.

We now investigate the case where θ is a constant at a point other than at the equilibrium point where $Z\neq Z_d$. Let this constant be θ_C. Then Eqs. (3-5) simplify to

$$u-\sum_{i=1}^{4}(rS_{i0}-M_{i0})=0 \tag{13}$$

$$-(M_{i0}-rS_{i0})=\int_r^l \rho x\frac{\partial^2 y_i}{\partial t^2}\,\mathrm{d}x+ml\left.\frac{\partial^2 y_i}{\partial t^2}\right|_l, \quad i=1,2,3,4 \tag{14}$$

$$\rho\frac{\partial^2 y_i}{\partial t^2}+EI\frac{\partial^4 y_i}{\partial x^4}=0, \qquad i=1,2,3,4 \tag{15}$$

The boundary conditions given by Eqs. (6) and (7) remain unchanged, but the boundary condition given by Eq. (8) simplifies to

$$\left.\frac{\partial^3 y_i}{\partial x^3}\right|_l=\frac{m}{EI}\left.\frac{\partial^2 y_i}{\partial t^2}\right|_l, \qquad i=1,2,3,4 \tag{16}$$

Also, the input to the system $u(t)$ defined by Eq. (11) can be simplified, using Eq. (13), to

$$u=\sum_{i=1}^{4}(rS_{i0}-M_{i0})=\frac{a_2}{a_3}(\theta_f-\theta_C)\triangleq C=\text{const} \tag{17}$$

If we define $Y=\sum_{i=1}^{4} y_i$, then Eq. (17) implies

$$\left[r\frac{\partial^3 Y}{\partial x^3}-\frac{\partial^2 Y}{\partial x^2}\right]_{x=r}=\frac{C}{EI}=\text{const} \tag{18}$$

If we make the reasonable assumption that $Y(x,t)$ is of the form $Y(x,t)=F(x)G(t)$, then Eq. (18) leads to

$$G(t)\left[r\frac{\partial^3 F}{\partial x^3}-\frac{\partial^2 F}{\partial x^2}\right]_{x=r}=\text{const} \tag{19}$$

Equation (19) implies that $G(t)$ is a constant. Summing Eqs. (15) and (16) over $i=1$ to $i=4$, we have

$$\rho\frac{\partial^2 Y}{\partial t^2}+EI\frac{\partial^4 Y}{\partial x^4}=0 \tag{20}$$

$$\left.\frac{\partial^3 Y}{\partial x^3}\right|_l=\frac{m}{EI}\left.\frac{\partial^2 Y}{\partial t^2}\right|_l \tag{21}$$

Because $Y(x,t)=F(x)G(t)$, and $G(t)$ is a constant, Eqs. (20) and (21) imply

$$\frac{\partial^4 Y}{\partial x^4}=0 \Rightarrow \frac{\partial^3 Y}{\partial x^3}=\text{const} \tag{22}$$

$$\left.\frac{\partial^3 Y}{\partial x^3}\right|_l=0 \tag{23}$$

From Eqs. (22) and (23) it follows that $(\partial^3 Y/\partial x^3)=0$, which implies that $(\partial^2 Y/\partial x^2)$ is a constant. Additionally, the value of this constant can be shown to be zero from the boundary condition in Eq. (7). Proceeding in the same way and using the boundary conditions in Eq. (6), it is trivial to show that $(\partial Y/\partial x)=Y(x,t)=0$. This implies from Eqs. (18) and (17) that $u=0$ and $\theta_C=\theta_f$. Clearly, the maximum invariant set for the system comprises the set of points where $\theta=\theta_f$, $\dot{\theta}=0$, and $\Sigma_{i=1}^4 y_i(x,t)=0$. If there exist functions $y_i(x,t)\neq 0$, $i=1,2,3,4$ such that $Y=\Sigma_{i=1}^4 y_i=0$ holds, then the set $S=\{Z: V^{(j)}(Z)=0, \forall j=1,2,\ldots,\infty\}$ contains entries other than the trivial solution $Z=Z_{\text{desired}}$. In such a situation we cannot claim global asymptotic stability of the equilibrium point. Such a situation may arise in the case of symmetric deformation in opposition, shown in Fig. 2, where $y_1(x,t)=-y_2(x,t)$ and $y_3(x,t)=-y_4(x,t)$. In such a situation, the residual energy of the system remains trapped within the beams. There exists no net interacting moment between the hub and the beams, and the hub remains motionless at its desired configuration $\theta=\theta_f$.

The case of antisymmetric deformation in unison, shown in Fig. 2, was considered in Ref. 4. In this case, it is assumed that $y_1(x,t)=y_2(x,t)=y_3(x,t)=y_4(x,t)$. When $Y(x,t)=0$, this implies that $y_i(x,t)=0$ for $i=1,2,3,4$. Therefore, for antisymmetric deformation in unison, it is quite simple to show that the set $S=\{Z: V^{(j)}(Z)=0, \forall j=1,2,\ldots,\infty\}$ contains only the equilibrium point $Z=Z_{\text{desired}}$. Consequently, we can establish the asymptotic stability property of the hub with the flexible appendages undergoing antisymmetric deformation in unison under the input defined by Eq. (11). The control law given in Eq. (11) was used to stabilize the system to the equilibrium point in Ref. 4, but no formal proof for the asymptotic stability was provided.

Conclusion

The rest-to-rest maneuver of the distributed parameter system consisting of a rigid hub with four cantilevered flexible appendages was studied. The best choice of the control input resulted in the negative semidefiniteness of the derivative of the Lyapunov function. An invariant set analysis of the system was subsequently carried out using an asymptotic stability theorem.[1] The analysis establishes the fact that the hub-appendage system is globally asymptotically stable when the system undergoes antisymmetric motion with deformation in unison.

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